Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 7, Number 3, July 2023
DOI : https://doi.org/10.33395/sinkron.v8i3.12744
e-ISSN : 2541-2019
p-ISSN : 2541-044X

A Class of Primitive Two-Colored Digraph with Large Competition Index

Ema Sri Rezeki[1)*], **Saib Suwilo**[2)], **Mardiningsih**[3)]
[1)]Graduated of Mathematics, Universitas Sumatera Utara, Indonesia
[2,3)]Department of Mathematics, Universitas Sumatera Utara, Indonesia
[1)]emasrirezeki13math@gmail.com, [2)]saib@usu.ac.id, [3)]mardiningsih.math@gmail.com

Submitted: Jul 22, 2023 | **Accepted** : Jul 23, 2023 | **Published** : Jul 23, 2023

Abstract: The competition index of a primitive two-colored digraph D^2, denoted $k(D^{(2)})$, is the smallest positive integer $h + l$ such that for each pair of vertices u and v there is vertex w with the property that there is a (h, l)-walk from v to w. For two-colored digraph on n vertices it is known that $k(D^{(2)}) \le (3n^3 + 2n^2 - 2n)/2$. In this work, we discuss a class of primitive two-colored digraph consisting of two cycles whose scrambling index closes to $(3n^3 + 2n^2 - 2n)/2$.

Keywords: Digraph, primitive digraphs, primitive two-colored digraph D^2

INTRODUCTION

Given D be a digraph. We follow digraph terminologies and two-colored digraphs from (Akelbek & Kirkland, 2009a, 2009b). A walk is a vertex u to a vertex v stated by $u \to v$ walk. That matter is a sequence of from the arcs, $u \to v_1, v_1 \to v_2, \dots, v_{k-1} \to v$. A $u \to v$ walk of length k is indicated by $u \overset{k}{\to} v$ walk. The length of a shortest $u \to v$ path is denoted by $d(u, v)$ which is a distance from a vertex u to vertex v. Suppose that E is a primitive strongly connected digraph then there is a positive integer k such that for each pair of vertices u and v there exists a $v_i \overset{k}{\to} v_j$ walk.

A digraph D each of its arcs is colored by either red or blue called a two-colored digraph $D^{(2)}$. The term of the positive integers h and l, an (h, l)-walk in a two-colored digraph $D^{(2)}$ is a walk including from h red arcs and l blue arcs. An (h, l)-walk from v_t to v_j is also symbolized by $v_i \overset{(h,l)}{\longrightarrow} v_j$ walk. A walk S in $(D^{(2)})$ is written by $r(S)$ and $b(S)$ to be the number of red arcs and blue arcs of S, respectively. The vector of the composition of S can be denoted by $(r(S), b(S))^T$. Let nonnegative integer h and l, a two-colored digraph $(D^{(2)})$ is said to be primitive if it is for each pair of vertices u and v in $(D^{(2)})$ there exist an $u \overset{h,l}{\to} v$ walk and an $v \overset{h,l}{\to} u$. The exponent of $D^{(2)}$ is the smallest positive integer $h + l$ over all nonnegative integers h and l. In 1998, (Fornasini & Valcher, 1998b, 1998a) reveal that a two-colored digraph $D^{(2)}$ is primitive if and only if the content of M is 1.

On the other side, (Shader & Suwilo, 2003; Suwilo & Shader, 2006) generalized the concept of exponent of two-colored digraphs. They examined that if n is odd, then the largest exponent of a two-colored digraph on n vertices lie on the interval $[\frac{n^3}{2}, (3n^3 + 2n^2 - 2n)]$. In the present paper, for some even integers $n \ge 5$, we will discuss the competition index of a class of primitive two-colored digraphs ${E_n}^{(2)}$ which consist of the cycle $v_1 \to v_2 \to \dots . \to v_{n-1} \to v_n \to v_1$ of length $n - 1$ and the cycle $v_1 \to v_2 \to v_3 \to \dots . \to v_{n-4} \to v_{n-3} \to v_1$ of length $n - 3$ as shown in Fig. 1, as follows

*name of corresponding author

e-ISSN : 2541-2019
p-ISSN : 2541-044X

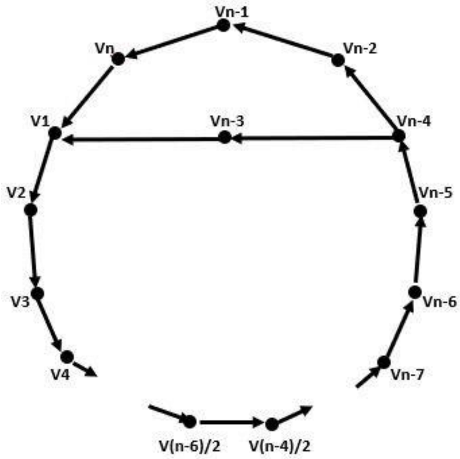

Fig. 1 Digraph E_n

We set the paper as follows, discussing primitivity of two-colored digraphs and presenting several results on bounds of competition index.

METHOD

To construct a class of primitive two-colored digraphs with a large competition index, we will devise a step-by-step approach. First, we define the class of digraphs to be created. Our foundation will consist of directed cycles, which are cyclic structures where each vertex has an outgoing arc to the next vertex in the cycle. For clarity, we will distinguish arcs with two colors: "red" and "blue." The objective is to create a class of digraphs that maximizes the competition index, which measures the size of the largest set of vertices where each vertex in the set has an arc to every vertex outside it (Mahlmann & Schindelhauer, 2006).

The construction begins by identifying certain parameters. One key parameter is the length of the directed cycle(s), which we denote as "k." The length of a directed cycle corresponds to the number of vertices in the cycle. Additionally, we introduce another parameter, "n," representing the number of cycles we will have in our digraphs. By varying "k" and "n," we can create digraphs of varying complexity.

Now, we proceed to construct the digraphs with a large competition index (Kim & Lee, 2012). First, we create "n" directed cycles, each with "k" vertices. These directed cycles will serve as the core building blocks of our digraphs. Next, we establish connections between the vertices in the different cycles. To achieve a large competition index, we ensure that no two vertices in different cycles share the same color of the arc between them (Valdes et al., 1979). For example, if we have two cycles, A and B, we can connect the vertices in A to the vertices in B using red arcs, and vice versa, using blue arcs.

By linking the cycles together in this manner, we guarantee a competition index of at least "k" because each cycle contributes at least "k" vertices to the competition set, and no vertex within a cycle can reach a vertex outside that cycle. Consequently, the size of the competition index grows significantly as we increase the number of cycles or the length of the directed cycles.

Furthermore, we want our digraphs to be primitive, meaning that for any pair of vertices, there exists a directed path from one vertex to the other. By connecting all cycles together, we ensure that there is a simple path from any vertex to any other vertex in the digraph, satisfying the primitivity condition (Akelbek & Kirkland, 2009b; Hussein et al., 2014).

In summary, our class of primitive two-colored digraphs with a large competition index consists of directed cycles interconnected in such a way that each cycle contributes to the competition set, and no two vertices in different cycles share the same color of the arc between them. This construction ensures a significant competition index, and the primitivity property guarantees the existence of a directed path between any two vertices in the digraphs. The actual size of the competition index depends on the chosen values of "k" and "n," providing flexibility to tailor the digraphs' characteristics to meet specific requirements. However, for exceptionally large competition indices, more complex constructions or techniques might be necessary.

*name of corresponding author

SinkrOn
JURNAL & PENELITIAN TEKNIK INFORMATIKA
Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 7, Number 3, July 2023
DOI : https://doi.org/10.33395/sinkron.v8i3.12744
e-ISSN : 2541-2019
p-ISSN : 2541-044X

RESULT

A $D^{(2)}$ digraph be a primitive two-colored digraph

If $D^{(2)}$ is a primitive of two-colored digraphs composed of two cycles C_1 and C_2, then the cycle matrix of $D^{(2)}$ is:

$$M = \begin{bmatrix} r(C_1) & r(C_2) \\ b(C_1) & b(C_2) \end{bmatrix},$$

Theorem 2.1. Let $D^{(2)}$ is a primitive two-colored digraph composed of two cycles with at least one arc for each color and assume M be a cycle matrix of $D^{(2)}$. Since two-colored digraph $D^{(2)}$ is a primitive if and only if $\det(M) = \pm 1$ (Akelbek & Kirkland, 2009b).

Bound on competition index

Proposition 2.2. If for any vertices u and v in $D^{(2)}$ is a primitive then we have $k_{u,v}(D^{(2)}) \leq k(D^{(2)}) \leq expin_{D^{(2)}}(u)$ (Shader & Suwilo, 2003).

The following result provide a way in decision a lower bound of competition index.

Lemma 2.3. Suppose $D^{(2)}$ is a primitive two-colored digraph composed of two-cycles C_1 and C_2 and assume M is a cycle matrix of $D^{(2)}$ with $\det(M) = \pm 1$. If $k_{v_i,v_j}(v_t)$ is given by (h, l)-walk, then,

$$\begin{bmatrix} h \\ k \end{bmatrix} \geq M \begin{bmatrix} b(C_2)r(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t}) \\ r(C_1)b(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t}) \end{bmatrix}$$

For some path P_{v_i,v_t} and P_{v_j,v_t}. Indeed,

$$k_{v_i,v_j}(v_t) \geq l(C_1)[b(C_2)r(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t}] + [r(C_1)b(P_{v_i,v_t}) - r(C_2)b(P_{v_i,v_t})]$$

In this section, we divide the result in the two parts. The first, we examine primitivity of two-colored digraphs and the second we present several results on bounds of competition index.

Theorem 3.1. Let $D^{(2)}$ be a two-colored digraph consisting of two cycles C_1 and C_2 with $l(C_1) = n - 3$ and $l(C_2) = n - 1$ (Gao & Shao, 2005). The two-colored $D^{(2)}$ is primitive if only if:

$$M = \begin{bmatrix} (n-2)/2 & n/2 \\ (n-1)/2 & (n-2)/2 \end{bmatrix}$$

The following theorem, Let $D^{(2)}$ be a two-colored digraph on even $n \geq 5$ vertices whose uncolored digraph as shown in Figure.1 is the digraph consisting of the cycle $C_1: v_1 \to v_2 \to v_3 \to \ldots. \to v_{n-4} \to v_{n-3} \to v_1$of length $l(C_1) = n - 3$ and the cycle $C_2: v_1 \to v_2 \to \ldots. \to v_{n-1} \to v_n \to v_1$ of length $l(C_2) = n - 1$. There are six characterizes of them as follows.

(i) The two-colored digraph $D^{(2)}$ is of type I if the blue arcs of $D^{(2)}$ are the arc that lie on the path $v_{n/2} \to v_{(n+2)/2} \to \cdots \to v_{n-1} \to v_n$ of length $(n-2)/2$ plus the arcs $v_{n-4} \to v_{n-3} \to v_1$.

(ii) The two-colored digraph $D^{(2)}$ is of type II if the blue arcs of $D^{(2)}$ are the arc that lie on the path $v_{(n+6)/2} \to v_{(n+8)/2} \to \cdots \to v_1 \to v_{(n-6)/2}$ of length $(n-2)/2$ plus the arcs $v_{n-4} \to v_{n-3} \to v_1$.

(iii) The two-colored digraph $D^{(2)}$ is of type III if the blue arcs of $D^{(2)}$ are the arc that lie on the path $v_{(n-4)/2} \to v_{(n-2)/2} \to \cdots \to v_{n-4} \to v_{n-2}$ of length $(n-2)/2$.

(iv) The two-colored digraph $D^{(2)}$ is of type IV if the blue arcs of $D^{(2)}$ are the arc that lie on the path $v_n \to v_1 \to \cdots \to v_{(n-4)/2} \to v_{(n-2)/2}$ of length $(n-2)/2$.

*name of corresponding author

SinkrOn
JURNAL & PENELITIAN TEKNIK INFORMATIKA
Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 7, Number 3, July 2023
DOI : https://doi.org/10.33395/sinkron.v8i3.12744
e-ISSN : 2541-2019
p-ISSN : 2541-044X

(v) The two-colored digraph $D^{(2)}$ is of type V if the blue arcs of $D^{(2)}$ are the arc that lie on the path $v_{(n-2)/2} \rightarrow v_{n/2} \rightarrow \cdots \rightarrow v_{n-2} \rightarrow v_{n-1}$ of length $(n-2)/2$ plus $v_{n-4} \rightarrow v_{n-3}$.

(vi) The two-colored digraph $D^{(2)}$ is of type VI if the blue arcs of $D^{(2)}$ are the arc that lie on the path $v_{n-1} \rightarrow v_n \rightarrow \cdots \rightarrow v_1 \ldots \rightarrow v_{(n-6)/2} \rightarrow v_{(n-4)/2}$ of length $(n-2)/2$ plus $v_{n-3} \rightarrow v_1$.

Theorem 3.2. Let $D^{(2)}$ be a two-colored digraphs consisting of type I on $n \geq 5$ vertices whose underlying digraph of both cycles (Syahmarani & Suwilo, 2012). Then $k\left(D^{(2)}\right) = (n^3 - 5n^2 + 4n + 4)/4$.

Proof. We show that $k(D^{(2)}) \geq (n^3 - 5n^2 + 4n + 4)/4$ by showing that $k_{v_n, v_{\frac{n}{2}}}(D^{(2)}) \geq (n^3 - 5n^2 + 4n + 4)/4$. Assume that there are $v_n \xrightarrow{(h,l)} w$ and $v_{n/2} \xrightarrow{(h,l)} w$ walks for some $w \in V(D^{(2)})$. We consider two cases depending on the position of the vertex w.

Case 1. $w = v_t; 1 \leq t \leq n/2$, there is a unique P_{nt} path from v_n to v_t which is a $(t, 0)$-path. Using this path and Lemma 2.3, we conclude that,

$$\begin{aligned} \boldsymbol{q_1} &\geq b(C_2)r(P_{nt}) - r(C_2)r(P_{nt}) \\ &= \frac{n-2}{2}(t) - \frac{n}{2}0 = \frac{n-2}{2}t \end{aligned}$$

There are two paths $P_{\frac{n}{2},t}$ from $v_{n/2}$ to v_t. They are $(t-1, (n-4)/2)$-path and $(t, (n-2)/2)$-path. Using the $(t-1, (n-4)/2)$-path and Lemma 2.3, we find that,

$$\begin{aligned} \boldsymbol{q_2} &\geq r(C_1)b\left(P_{n/2,t}\right) - b(C_1)r(P_{n/2,t}) \\ &= \frac{n-2}{2}\frac{n-4}{2} - \frac{n-4}{2}(t-1) = \frac{n^2-4n}{4} - \frac{n-4}{2}t. \end{aligned}$$

Using the $(t, (n-2)/2)$-path and Lemma 2.3, we find that,

$$\begin{aligned} \boldsymbol{q_2} &\geq r(C_1)b\left(P_{n/2,t}\right) - b(C_1)r(P_{n/2,t}) \\ &= \frac{n-2}{2}\frac{n-2}{2} - \frac{n-4}{2}t = \frac{n^2-4n+4}{4} - \frac{n-4}{2}t. \end{aligned}$$

Hence, we conclude that $q_2 \geq \frac{n^2-4n}{4} - \frac{n-4}{2}t$. By considering Lemma 2.2, we conclude that,

$$\begin{bmatrix} h \\ k \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{n^3-4n^2}{8} + t \\ \frac{n^3-6n^2+8n}{8} \end{bmatrix} \geq \begin{bmatrix} \frac{n^3-4n^2+8}{8} \\ \frac{n^3-6n^2+8n}{8} \end{bmatrix}$$

Since $t \geq 1$.

Case 2. $w = v_t, \frac{n+2}{2} \leq t \leq n.$

There is a unique path from v_n to v_t which is a $(\frac{n}{2}, t - \frac{n}{2})$-path. Using this path and Lemma 2.3, we conclude that,

$$\begin{aligned} \boldsymbol{q_1} &\geq b(C_2)r(P_{nt}) - r(C_2)r(P_{nt}) \\ &= \frac{n-2}{2}\frac{n}{2} - \frac{n}{2}(t - \frac{n}{2}) \end{aligned}$$

There is a unique path $P_{\frac{n+2}{2},t}$ from $v_{\frac{n+2}{2}}$ to v_t. They are $(0, t - \frac{n}{2})$-path. Using the $(0, t - \frac{n}{2})$-path and Lemma 2.3, we find that,

$$\begin{aligned} \boldsymbol{q_2} &\geq r(C_1)b\left(P_{\frac{n+2}{2},t}\right) - b(C_1)r(P_{\frac{n+2}{2},t}) \\ &= \frac{n-2}{2}(t - \frac{n}{2}) \end{aligned}$$

*name of corresponding author

SinkrOn
JURNAL & PENELITIAN TEKNIK INFORMATIKA
Sinkron : Jurnal dan Penelitian Teknik Informatika
Volume 7, Number 3, July 2023
DOI : https://doi.org/10.33395/sinkron.v8i3.12744
e-ISSN : 2541-2019
p-ISSN : 2541-044X

By Lemma 2.2, we conclude that,

$$\begin{bmatrix} h \\ k \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \dfrac{n^3-4n^2+4n}{8} \\ \dfrac{n^3-6n^2+4n}{8}+t \end{bmatrix} \geq \begin{bmatrix} \dfrac{n^3-4n^2+4n}{8} \\ \dfrac{n^3-6n^2+8n+8}{8} \end{bmatrix}$$

Since $t \geq \frac{n+2}{2}$.

Now from case 1 and case 2 we conclude that,

$$\begin{bmatrix} h \\ k \end{bmatrix} \geq \begin{bmatrix} \dfrac{n^3-4n^2+8}{8} \\ \dfrac{n^3-6n^2+8n}{8} \end{bmatrix}$$

This implies $k_{v_n,v_{\frac{n}{2}}}(D^{(2)}) \geq \frac{n^3-5n^2+4n+4}{4}$ and hence $k(D^{(2)}) \geq \frac{n^3-5n^2+4n+4}{4}$.

We know show that $k(D^{(2)}) \geq \frac{n^3-5n^2+4n+4}{4}$ by showing that for each vertex $v_{t,}t = 1,2,3 \dots, n$ there is an (h,l)-walk from v_t to v_1 with

$$\begin{bmatrix} h \\ k \end{bmatrix} \geq \begin{bmatrix} \dfrac{n^3-4n^2+8}{8} \\ \dfrac{n^3-6n^2+8n}{8} \end{bmatrix}$$

For this purpose, we consider the system of equation:

$$Mz + \begin{bmatrix} r(p_{t1}) \\ b(p_{t1} \end{bmatrix} = \begin{bmatrix} r(P_{t1}) \\ b(P_{t1} \end{bmatrix}$$

And show this system has nonnegative integer solution for some path P_{t1} from v_t to v_1. The solution of the system is the integer vector.

$$z = \begin{bmatrix} \dfrac{n-2}{2} + \dfrac{n}{2} b(P_{t1}) - \dfrac{n-2}{2} r(P_{t1}) \\ \dfrac{n-4}{2}\dfrac{n-2}{2} + \dfrac{n-4}{2} r(P_{t1}) - \dfrac{n-2}{2} b(P_{t1}) \end{bmatrix}$$

If $1 \leq t \leq \frac{n}{2}$, then there is a $(\frac{n}{2}-t, \frac{n-4}{2})$-path from v_t to v_1. Using this path, we have $z_1 = \frac{n}{2}t - (1 + t)$ and $z_2 = \frac{n-4}{2}(\frac{n-2}{2} + (1-t))$. Since $t \geq 1$ we have $z_1 > 0$ and since $t \leq \frac{n}{2}$, we have $z_2 \geq 0$. If $\frac{n+2}{2} \leq t \leq n-4$. Then there is a $(0, n-2-t)$-path from v_t to v_1. Using this path, we have z_1.

Theorem 3.3. Let $D^{(2)}$ be a two-colored digraphs consisting of type II on $n \geq 5$ vertices whose underlying digraph of both cycles. Then $k(D^{(2)}) = (n^3 - 5n^2 + 6n + 8)/4$.

Theorem 3.4. Let $D^{(2)}$ be a two-colored digraphs consisting of type III on $n \geq 5$ vertices whose underlying digraph of both cycles. Then $k(D^{(2)}) = (n^3 - 5n^2 + 4n + 8)/4$.

Theorem 3.5. Let $D^{(2)}$ be a two-colored digraphs consisting of type IV on $n \geq 5$ vertices whose underlying digraph of both cycles. Then $k(D^{(2)}) = (n^3 - 5n^2 + 6n + 4)/4$.

Theorem 3.6. Let $D^{(2)}$ be a two-colored digraphs consisting of type V on $n \geq 5$ vertices whose underlying digraph of both cycles. Then $k(D^{(2)}) = (n^3 - 5n^2 + 4n + 4)/4$.

Theorem 3.7. Let $D^{(2)}$ be a two-colored digraphs consisting of type VI on $n \geq 5$ vertices whose underlying digraph of both cycles. Then $k(D^{(2)}) = (n^3 - 5n^2 + 6n + 8)/4$.

*name of corresponding author

DISCUSSIONS

In conclusion, we have successfully constructed a class of primitive two-colored digraphs with a large competition index. By combining directed cycles as the foundational building blocks and carefully interconnecting them, we ensured that each cycle contributes to the competition set, and no two vertices in different cycles share the same color of the arc between them. This design results in a competition index that is at least as large as the length of the directed cycles, making it significantly large.

The primitivity property of the digraphs ensures that there exists a directed path between any pair of vertices, making the digraphs well-connected and allowing for efficient traversal from one vertex to another. The flexibility in selecting the parameters "k" and "n" enables us to customize the digraphs to meet specific requirements. By adjusting these parameters, we can create digraphs with even larger competition indices, catering to different application scenarios and optimization goals.

Overall, the class of primitive two-colored digraphs with a large competition index has potential applications in various fields, including network design, optimization problems, and the study of complex systems. The richness of these digraphs opens up possibilities for further research and exploration in graph theory and related areas, making them valuable tools in understanding and analyzing complex structures and relationships.

REFERENCES

Akelbek, M., & Kirkland, S. (2009a). Coefficients of ergodicity and the scrambling index. *Linear Algebra and Its Applications*, *430*(4), 1111–1130.

Akelbek, M., & Kirkland, S. (2009b). Primitive digraphs with the largest scrambling index. *Linear Algebra and Its Applications*, *430*(4), 1099–1110.

Fornasini, E., & Valcher, M. E. (1998a). Multidimensional systems with finite support behaviors: Signal structure, generation, and detection. *SIAM Journal on Control and Optimization*, *36*(2), 760–779.

Fornasini, E., & Valcher, M. E. (1998b). Primitivity of positive matrix pairs: algebraic characterization, graph theoretic description, and 2D systems interpretation. *SIAM Journal on Matrix Analysis and Applications*, *19*(1), 71–88.

Gao, Y., & Shao, Y. (2005). Exponents of two-colored digraphs with two cycles. *Linear Algebra and Its Applications*, *407*, 263–276.

Hussein, I., Prasetyo, Y. D., & Suwilo, S. (2014). Upper generalized exponents of two-colored primitive extremal ministrong digraphs. *AIP Conference Proceedings*, *1635*(1), 430–439.

Kim, H. K., & Lee, S. H. (2012). Generalized competition indices of symmetric primitive digraphs. *Discrete Applied Mathematics*, *160*(10–11), 1583–1590.

Mahlmann, P., & Schindelhauer, C. (2006). Distributed random digraph transformations for peer-to-peer networks. *Proceedings of the Eighteenth Annual ACM Symposium on Parallelism in Algorithms and Architectures*, 308–317.

Shader, B. L., & Suwilo, S. (2003). Exponents of nonnegative matrix pairs. *Linear Algebra and Its Applications*, *363*, 275–293.

Suwilo, S., & Shader, B. L. (2006). On 2-exponents of ministrong 2-digraphs. *Pure Mathematics*, 51.

Syahmarani, A., & Suwilo, S. (2012). Vertex exponents of a class of two-colored hamiltonian digraphs. *Journal of the Indonesian Mathematical Society*, 1–19.

Valdes, J., Tarjan, R. E., & Lawler, E. L. (1979). The recognition of series parallel digraphs. *Proceedings of the Eleventh Annual ACM Symposium on Theory of Computing*, 1–12.

*name of corresponding author